On the center of affine Hecke algebras of type A

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0.1 Introduction. Let G be a simple complex algebraic group. Let W be its Weyl group and $\widehat{W}$ the associated extended affine Weyl group. Let $\widehat{\mathbf{H}}$ be the Iwahori-Hecke algebra of $\widehat{W}$. It is well-known that $\widehat{\mathbf{H}}$ admits two presentations : the *Coxeter* presentation which arises naturally when $\widehat{\mathbf{H}}$ is realized as the convolution algebra $L(\widehat{G}, I)$ of compactly supported functions on a p-adic group $\widehat{G} = G(\overline{\mathbb{Q}_p})$ which are bi-invariant under action of the Iwahori subgroup I (see [IM]), and the *Bernstein* presentation, which arises when $\widehat{\mathbf{H}}$ is realized in the $G^\vee \times \mathbb{C}^*$-equivariant K-theory of the Steinberg variety associated to $G^\vee$ where $G^\vee$ is the Langlands dual group (see [Gi]). The interplay between these two presentations is central in the Deligne-Langlands correspondence for finite-dimensional irreducible representations of $\widehat{\mathbf{H}}$.

The center $Z(\widehat{\mathbf{H}})$ is easily described in the K-theoretic picture : it is spanned by the classes of the trivial (equivariant) bundles on Z. A geometric construction of this center in the convolution algebra presentation is given by Gaitsgory, [Ga]. This is in turn inspired by work of Beilinson, and Haines, Kottwitz and Rapoport in the framework of Shimura varieties, see [H1],[H2].

In this paper we give an explicit expression for the central elements of $\widehat{\mathbf{H}}$ in the Coxeter presentation when $G = GL(r)$ (Theorem 2.5). This expression generalizes those obtained by Haines in the minuscule case, [H2] and is in some sense more explicit than [Ga]. More generally, we obtain expressions for central elements in the "parabolic spherical" Hecke algebras $L(\widehat{G}, P)$ where $P \supseteq I$ is a parahoric subgroup. In particular, taking $P = K$ to be a maximal compact open subgroup recovers Lusztig's description [L1] of the Satake isomorphism between $Z(\widehat{\mathbf{H}})$ and the spherical algebra $\widehat{\mathbf{H}}_{sph}$ (in the case $G = GL(r)$).

Our method is based on the Hall algebra of a cyclic quiver, on Uglov's higher-level Fock spaces and on the theory of canonical bases of Kashiwara and Lusztig. Namely, we use Ginzburg and Vasserot's geometric description of quantum affine Schur-Weyl duality to construct an embedding of (half of) the center $Z(\widehat{\mathbf{H}})$ in the center of the Hall algebra $\mathbf{U}_n^-$ of the quiver $\widetilde{A}_{n-1}$ for $n \geq r$ (see [S]). This embedding is compatible with the canonical bases of $\widehat{\mathbf{H}}$ and $\mathbf{U}_n^-$. To describe the center of $\mathbf{U}_n^-$ we then consider the action on the Fock spaces $\Lambda^\infty_{\mathbf{s}_l}$ recently introduced by Uglov [U], and use the fact that this action is again compatible with the canonical bases.

Finally, we give a simple alternate description of the center of $\mathbf{U}_n^-$ in terms of a certain desingularization of orbit closures of representations of the quiver

$\widetilde{A}_{n-1}$, introduced by Varagnolo and Vasserot [VV]. This can be seen as a cyclic analogue of the desingularization of orbit closures recently obtained by Reineke [Re] for finite-type simply laced Dynkin quivers.

We note that the Fock spaces and their canonical bases appear to be a very fundamental object in type A representation theory : they describe Grothendieck groups and decomposition numbers of Hecke algebras of type A or B (or more generally cyclotomic Hecke algebras) at roots of unity (see [LLT],[A], [AM], [Gro]), and modular representations of symmetric groups (see [Di], [J], [Gro]).

0.2 Notations. Set $\mathbb{S} = \mathbb{C}[v]$, $\mathbb{A} = \mathbb{C}[v, v^{-1}]$ and $\mathbb{K} = \mathbb{C}(v)$. We define a $\mathbb{C}$-linear ring involution $u \mapsto \overline{u}$ on $\mathbb{A}$ by setting $\overline{v} = v^{-1}$. Let $\mathbb{F}$ be a finite field with q^2 elements. Let $\mathfrak{S}_r$ denote the symmetric group on r elements and let $\{s_i\}_{i=1,\dots r-1}$ be the set of simple reflections. Let $\widehat{\mathfrak{S}}_r = \mathfrak{S}_r \ltimes \mathbb{Z}^r$ be the *extended* affine symmetric group and let s_0 be the affine simple reflection. Let Π stand for the set of partitions and let Π_r be the set of partitions of length at most r. Elements of Π^l for some $l \in \mathbb{N}$ will be called *l-multipartitions*. Finally, we will denote by $\overline{Y}$ the Zariski closure of any subset Y of an algebraic variety X.

1 Affine Hecke algebras and canonical bases

1.1 Consider the Iwahori-Hecke algebra $\widehat{\mathbf{H}}_r$ associated to $\widehat{\mathfrak{S}}_r$, i.e the $\mathbb{A}$-algebra generated by elements T_σ, $\sigma \in \widehat{\mathfrak{S}}_r$ with relations

$$(T_{s_i} + 1)(T_{s_i} - v^{-2}) = 0 \qquad for\ i = 0, \dots, r-1,$$

$$T_\sigma T_\gamma = T_{\sigma\gamma} \qquad if\ l(\sigma\gamma) = l(\sigma) l(\gamma).$$

We set $\tilde{T}_\sigma = v^{l(\sigma)} T_\sigma$ for every $\sigma \in \widehat{\mathfrak{S}}_r$.

It is well-known that $\widehat{\mathbf{H}}_r$ admits another presentation (the *Bernstein* presentation) as the unital $\mathbb{A}$-algebra generated by elements $T_i^{\pm 1}, X_j^{\pm 1}$ where $i \in [1, r-1]$, $j \in [1, r]$ with the following relations

$$\begin{array}{ll} T_i\, T_i^{-1} = 1 = T_i^{-1}\, T_i, & (T_i + 1)(T_i - v^{-2}) = 0, \\ T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1}, & |i-j| > 1 \Rightarrow T_i\, T_j = T_j\, T_i, \\ X_i\, X_i^{-1} = 1 = X_i^{-1}\, X_i, & X_i\, X_j = X_j\, X_i, \\ T_i\, X_i\, T_i = v^{-2} X_{i+1}, & j \neq i, i+1 \Rightarrow X_j\, T_i = T_i\, X_j. \end{array}$$

The isomorphism between the two presentations is such that $T_{s_i} \mapsto T_i$ and $\tilde{T}_\lambda^{-1} \mapsto X_1^{\lambda_1} \cdots X_r^{\lambda_r}$ if $\lambda = (\lambda_1, \dots, \lambda_r)$ is *dominant*. The center of $\widehat{\mathbf{H}}_r$ is $Z(\widehat{\mathbf{H}}_r) = \mathbb{A}[X_1^{\pm 1}, \dots, X_r^{\pm 1}]^{\mathfrak{S}_r}$. Set $Z_r^- = \mathbb{A}[X_1^{-1}, \dots, X_r^{-1}]^{\mathfrak{S}_r}$.

1.2 For every $t, s \in \mathbb{N}$ define the left (resp. right) representation of $\widehat{\mathfrak{S}}_t$ on $\mathbb{Z}^t$ of *level* s by

$$s_j \cdot (i_1, \dots, i_t) = (i_1, \dots, i_{j+1}, i_j, \dots i_t), \qquad 1 \leq j < r,$$

$$\lambda \cdot (i_1, \dots, i_t) = (i_1 + s\lambda_1, \dots, i_t + s\lambda_t), \qquad \lambda \in \mathbb{Z}^t$$

and

$$(i_1, \dots, i_t) \cdot s_j = (i_1, \dots, i_{j+1}, i_j, \dots i_t), \qquad 1 \leq j < r,$$
$$(i_1, \dots, i_t) \cdot \lambda = (i_1 + s\lambda_1, \dots, i_t + s\lambda_t), \qquad \lambda \in \mathbb{Z}^t$$

respectively. The set $\mathcal{A}_t^s = \{1 \leq i_1 \leq \cdots \leq i_t \leq s\}$ is a fundamental domain for both actions. For each $\mathbf{i} \in \mathcal{A}_t^s$ we set $\mathfrak{S}_{\mathbf{i}} = \mathit{Stab}\ \mathbf{i} \subset \mathfrak{S}_t$ and denote by $\omega_{\mathbf{i}} \in \mathfrak{S}_{\mathbf{i}}$ the longest element. We also let $\mathfrak{S}^{\mathbf{i}}$ be the set of all minimal length elements of the cosets $\mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_t$.

1.3 Fix some $n \in \mathbb{N}^*$. For any $\mathbf{i}, \mathbf{j} \in \mathcal{A}_r^n$ and any $\sigma \in \mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_r / \mathfrak{S}_{\mathbf{j}}$ we set $T_\sigma = \sum_{\delta \in \sigma} T_\delta$ and we let $\widehat{\mathbf{H}}_{\mathbf{ij}} \subset \widehat{\mathbf{H}}_r$ be the $\mathbb{A}$-linear span of the elements T_σ for $\sigma \in \mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_r / \mathfrak{S}_{\mathbf{j}}$. Set $e_{\mathbf{i}} = \sum_{\delta \in \mathfrak{S}_{\mathbf{i}}} T_\delta$. Then $\widehat{\mathbf{H}}_{\mathbf{ij}} = e_{\mathbf{i}} \widehat{\mathbf{H}}_r e_{\mathbf{j}}$. Put

$$\widehat{\mathbf{S}}_{n,r} = \bigoplus_{\mathbf{i}, \mathbf{j} \in \mathcal{A}_r^n} \widehat{\mathbf{H}}_{\mathbf{ij}}.$$

This space, equipped with the multiplication

$$e_{\mathbf{i}} h e_{\mathbf{j}} \bullet e_{\mathbf{k}} h' e_{\mathbf{l}} = \delta_{jk} e_{\mathbf{i}} h e_{\mathbf{j}} h' e_{\mathbf{l}} \in \widehat{\mathbf{H}}_{\mathbf{il}} \qquad \textit{for all } h, h' \in \widehat{\mathbf{H}}_r$$

is called the *affine q-Schur algebra*. It is proved in [GV], [L3] that $\widehat{\mathbf{S}}_{n,r}$ is a quotient of the *modified* quantum affine algebra $\dot{\mathbf{U}}_v^-(\widehat{\mathfrak{gl}}_n)$.

1.3 Set $\mathbf{T}_{n,r} = \bigoplus_{\mathbf{i} \in \mathcal{A}_r^n} e_{\mathbf{i}} \widehat{\mathbf{H}}_r$. For $\sigma \in \mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_r$ we put $T_\sigma = \sum_{\delta \in \sigma} T_\delta$. Then $\{\mathbf{T}_\sigma\}$, $\sigma \in \mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_r$ is an $\mathbb{A}$-basis of $e_{\mathbf{i}} \widehat{\mathbf{H}}_r$. It will be convenient to identify the element σ with $\mathbf{i} \cdot \sigma \in \mathbb{Z}^r$, so that $\{\mathbf{T}_p\}$, $p \in \mathbb{Z}^r$ is an $\mathbb{A}$-basis of $\mathbf{T}_{n,r}$.

The algebra $\widehat{\mathbf{H}}_r$ acts on $\mathbf{T}_{n,r}$ by multiplication on the right, and $\widehat{\mathbf{S}}_{n,r}$ acts on $\mathbf{T}_{n,r}$ on the left by

$$e_{\mathbf{i}} h e_{\mathbf{j}} \cdot e_{\mathbf{k}} h' = \delta_{\mathbf{jk}} e_{\mathbf{i}} h e_{\mathbf{j}} h' \in e_{\mathbf{i}} \widehat{\mathbf{H}}_r \qquad \textit{for every } h, h' \in \widehat{\mathbf{H}}_r.$$

Let us denote these actions by $\rho_r : \widehat{\mathbf{S}}_{n,r} \to \mathrm{End}\ (\mathbf{T}_{n,r})$ and $\sigma_r : \widehat{\mathbf{H}}_r \to \mathrm{End}\ (\mathbf{T}_{n,r})$. It is obvious that these two actions commute. The following result is a quantum and affine analogue of Schur-Weyl duality.

Theorem ([VV]). *We have* $\widehat{\mathbf{S}}_{n,r} = \mathrm{End}_{\widehat{\mathbf{H}}_r}(\mathbf{T}_{n,r})$. *Moreover, we have* $\widehat{\mathbf{H}}_r = \mathrm{End}_{\widehat{\mathbf{S}}_{n,r}}(\mathbf{T}_{n,r})$ *if* $n \geq r$.

1.4 Let us now, following [GV] and [IM], give the geometric realization of the above Schur-Weyl duality. Let $\mathbb{L} = \mathbb{F}((z))$ and set $\mathbb{G} = GL_r(\mathbb{L})$. By definition, a *lattice* in $\mathbb{L}^r$ is a free $\mathbb{F}[[z]]$-submodule of rank r. Consider the variety X of sequences of lattices $(L_i)_{i \in \mathbb{Z}}$ such that

$$L_i \subset L_{i+1}, \qquad \dim_{\mathbb{F}}(L_i / L_{i-1}) = 1, \qquad L_{i+r} = z^{-1} L_i$$

(the *affine flag variety of type* GL_r). Consider also the variety Y of all n-step periodic flags in $\mathbb{L}^r$, i.e the set of all sequences of lattices $(L_i)_{i \in \mathbb{Z}}$ such that

$$L_i \subset L_{i+1}, \qquad L_{i+n} = z^{-1} L_i$$

(the *affine partial flags variety*). The group $\mathbb{G}$ acts (transitively) on X and acts on Y in obvious ways. Consider the diagonal action of $\mathbb{G}$ on $X \times X$ and $Y \times Y$ respectively.

It is well-known that the set of $\mathbb{G}$-orbits on $X \times X$ is canonically identified with $\widehat{\mathfrak{S}}_r$. In order to describe these $\mathbb{G}$-orbits we let $(e_1, \dots, e_r)$ be a fixed $\mathbb{L}$-basis of $\mathbb{L}^r$ and set $e_{i+kr} = z^{-k} e_i$. Consider the right action of $\widehat{\mathfrak{S}}_r$ on $\mathbb{Z}^r$ of level r. To any element $\mathbf{x}$ in the orbit of $\rho_r = (1, 2, \dots, r)$ we associate the flag $(L(\mathbf{x})_i)_{i \in \mathbb{Z}}$ defined by

$$L(\mathbf{x})_i = \prod_{p(j) \leq i} \mathbb{F} e_j,$$

where $p : \mathbb{Z} \to \mathbb{Z}$ is the bijection uniquely defined by $p(j) = \mathbf{x}_j$ if $1 \leq j \leq r$ and $p(j + r) = p(j) + r$. The $\mathbb{G}$-orbit decomposition of $X \times X$ reads

$$X \times X = \bigsqcup_{\sigma \in \widehat{\mathfrak{S}}_r} X_\sigma$$

where $X_\sigma = \mathbb{G} \cdot (L(\rho_r \cdot \sigma), L(\rho_r))$. Similarly, to each $\mathbf{i} \in \mathbb{Z}^r$ we associate the map $p : \mathbb{Z} \to \mathbb{Z}$ uniquely defined by $p(j) = \mathbf{i}_j$ if $1 \leq j \leq r$ and $p(j + r) = p(j) + n$. Consider the flag

$$L(\mathbf{i})_i = \prod_{\mathbf{i}(j) \leq i} \mathbb{F} e_j.$$

Then $Y = \bigsqcup_{\mathbf{i} \in \mathcal{A}_r^n} Y_{\mathbf{i}}$ where $Y_{\mathbf{i}} = \mathbb{G} \cdot (L(\mathbf{i}))$ and

$$Y_{\mathbf{i}} \times Y_{\mathbf{j}} = \bigsqcup_{\sigma \in \mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_r / \mathfrak{S}_{\mathbf{j}}} Y_\sigma$$

where $Y_\sigma = \mathbb{G} \cdot (L(\mathbf{i} \cdot \sigma), L(\mathbf{j}))$ and where the right action of $\widehat{\mathfrak{S}}_r$ on $\mathbb{Z}^r$ is now of level n.

Let $\mathbb{C}_{\mathbb{G}}(X \times X)$ (resp. $\mathbb{C}_{\mathbb{G}}(Y \times Y)$) be the space of complex-valued $\mathbb{G}$-invariant functions on $X \times X$ (resp. on $Y \times Y$) which are supported on finitely many orbits. The convolution product endows these spaces with an associative algebra structure. We let $\mathbf{1}_{\mathcal{O}} \in \mathbb{C}_{\mathbb{G}}(X \times X)$ (resp. $\mathbf{1}_{\mathcal{O}} \in \mathbb{C}_{\mathbb{G}}(Y \times Y)$) be the characteristic function of a $\mathbb{G}$-orbit $\mathcal{O} \subset X \times X$(resp. $\mathcal{O} \subset Y \times Y$).

Theorem ([IM],[VV]).

i) The linear map $(\widehat{\mathbf{H}}_r)_{|v=q^{-1}} \to \mathbb{C}_{\mathbb{G}}(X \times X)$ defined by $T_\sigma \mapsto \mathbf{1}_{X_\sigma}$ is an algebra isomorphism.

ii) The linear map $(\widehat{S}_{n,r})_{|v=q^{-1}} \to \mathbb{C}_{\mathbb{G}}(Y \times Y)$ such that $T_\sigma \mapsto \mathbf{1}_{Y_\sigma}$ is an algebra isomorphism.

Now consider the diagonal action of $\mathbb{G}$ on $Y \times X$. The collection of orbits are parametrized by $\mathbb{Z}^r$: to $\mathbf{i} \in \mathbb{Z}^r$ corresponds the orbit $\mathcal{O}_{\mathbf{i}}$ of the pair $(L(\mathbf{i}), L(\rho_r))$. The algebras $\mathbb{C}_{\mathbb{G}}(X \times X)$ and $\mathbb{C}_{\mathbb{G}}(Y \times Y)$ act by convolution on $\mathbb{C}_{\mathbb{G}}(Y \times X)$ on the right and on the left respectively.

Theorem ([VV]). *The map $(\mathbf{T}_{n,r})_{|v=q^{-1}} \to \mathbb{C}_{\mathbb{G}}(Y \times X)$ such that $e_{\mathbf{i}} \mapsto \mathbf{1}_{\mathcal{O}_{\mathbf{i}}}$ for $\mathbf{i} \in \mathcal{A}_r^n$ extends uniquely to an isomorphism of $(\widehat{\mathbf{S}}_{n,r})_{|v=q^{-1}} \times (\widehat{\mathbf{H}}_r)_{|v=q^{-1}}$-modules.*

1.5 Let $u \mapsto \overline{u}$ be the semilinear involution of $\widehat{\mathbf{H}}_r$ defined by $\overline{T_\sigma} = T_{\sigma^{-1}}^{-1}$ for all σ. For each $\sigma \in \widehat{\mathfrak{S}}_r$ there exists a unique element $\mathbf{c}_\sigma \in \widehat{\mathbf{H}}_r$ such that

$$\text{i) } \overline{\mathbf{c}_\sigma} = \mathbf{c}_\sigma, \qquad \text{ii) } \mathbf{c}_\sigma = \tilde{T}_\sigma + \sum_{\delta<\sigma} c_{\delta,\sigma}(v)\tilde{T}_\delta, \quad c_{\delta,\sigma}(v) \in v\mathbb{S}.$$

The polynomial $c_{\sigma,\delta}(v)$ is the affine Kazhdan-Lusztig polynomial of type $\tilde{A}_{r-1}$ associated to σ and δ (this polynomial is denoted by $h_{\sigma,\delta}$ in Soergel's notation [Soe]).

For $\sigma \in \widehat{\mathfrak{S}}_r$ and $L \in X$ let $X_{\sigma,L}$ be the fiber of the first projection $X_\sigma \to X$. Then $X_{\sigma,L}$ is the set of $\mathbb{F}$-points of an algebraic variety of dimension $l(\sigma)$ whose isomorphism class is independent of L. Then

$$\mathbf{c}_\sigma = \sum_{i,\delta} v^{-i+l(\sigma)-l(\delta)} \dim \mathcal{H}^i_{X_{\delta,L}}(IC_{X_{\sigma,L}})\tilde{T}_\delta$$

where $IC_{X_{\sigma,L}}$ denotes the intersection cohomology complex associated to $X_{\sigma,L}$ and where $\mathcal{H}^i$ stands for local cohomology.

Similarly, let $\mathbf{i}, \mathbf{j} \in \mathcal{A}^n_r$ and let $\sigma \in \mathfrak{S}_\mathbf{i}\backslash\widehat{\mathfrak{S}}_r/\mathfrak{S}_\mathbf{j}$. Denote by $Y_{\sigma,\mathbf{i}}$ the fiber above $(L(\mathbf{i}))$ of the projection of $Y_\sigma \to Y$ on the first component. This is the set of $\mathbb{F}$-points of an algebraic variety of dimension, say $y(\sigma)$ (an explicit formula for $y(\sigma)$ can be found in [L3]). Put $\tilde{T}_\sigma = v^{y(\sigma)}T_\sigma$. For every $\sigma \in \mathfrak{S}_\mathbf{i}\backslash\widehat{\mathfrak{S}}_r/\mathfrak{S}_\mathbf{j}$ set

$$\mathbf{c}_\sigma = \sum_{i,\delta} v^{-i+y(\sigma)-y(\delta)} \dim \mathcal{H}^i_{Y_{\delta,\mathbf{i}}}(IC_{Y_{\sigma,\mathbf{i}}})\tilde{T}_\delta.$$

It is clear that $\overline{\widehat{\mathbf{H}}_{\mathbf{ij}}} = \widehat{\mathbf{H}}_{\mathbf{ij}}$. Define a semilinear involution $\tau : \widehat{\mathbf{H}}_{\mathbf{ij}} \to \widehat{\mathbf{H}}_{\mathbf{ij}}$ by $\tau(u) = v^{-2l(\omega_\mathbf{j})}\overline{u}$. The elements $\{\mathbf{c}_\sigma\}$ for all $\mathbf{i}, \mathbf{j} \in \mathcal{A}^n_r$ form the canonical basis of $\widehat{\mathbf{S}}_{n,r}$ and are characterized by the following two properties :

$$\text{i) } \tau(\mathbf{c}_\sigma) = \mathbf{c}_\sigma, \qquad \text{ii) } \mathbf{c}_\sigma = \tilde{T}_\sigma + \sum_{\delta<\sigma} c_{\delta,\sigma}(v)\tilde{T}_\delta, \qquad c_{\delta,\sigma}(v) \in v\mathbb{S}.$$

1.6 Let $s, t \in \mathbb{N}^*$. For $\mathbf{i} \in \mathcal{A}^s_t$ and $x \in \mathbf{i} \cdot \widehat{\mathfrak{S}}_t$ set $\langle x| = e_\mathbf{i}\tilde{T}_a$ where $\mathbf{i} \cdot a = x$ and $a \in \mathfrak{S}^\mathbf{i}$. The set $\{\langle x|, x \in \mathbf{i} \cdot \widehat{\mathfrak{S}}_t\}$ is an $\mathbb{A}$-basis of the space $e_\mathbf{i}\widehat{\mathbf{H}}_t$. Define a semilinear involution $u \mapsto \overline{u}$ of $e_\mathbf{i}\widehat{\mathbf{H}}_t$ by $\overline{e_\mathbf{i}x} = e_\mathbf{i}\overline{x}$. There exists a unique $\mathbb{A}$-basis $\{\mathbf{c}^-_x, x \in \mathbf{i} \cdot \widehat{\mathfrak{S}}_t\}$ of $e_\mathbf{i}\widehat{\mathbf{H}}_t$ such that

$$\text{i) } \overline{\mathbf{c}^-_x} = \mathbf{c}^-_x, \qquad \text{ii) } \mathbf{c}^-_x = \langle x| + \sum_y P^-_{y,x}\langle y|, \qquad P^-_{y,x} \in v^{-1}\mathbb{Z}[v^{-1}].$$

The polynomials $P^-_{y,x}$ are *parabolic affine Kazhdan-Lusztig polynomials* introduced by Deodhar [De]. These polynomials are (up to a sign) denoted by $\overline{n}_{a_y,a_x}$ in Soergel's notation, where $a_x, a_y \in \mathfrak{S}^\mathbf{i}$ are such that $x = \mathbf{i} \cdot a_x$, $y = \mathbf{i} \cdot a_y$.

2 The main result

2.1 Let Γ be Macdonald's ring of symmetric polynomial in the variables y_i, $i \in \mathbb{Z}$, defined over $\mathbb{A}$ (see [Mac]). Let $\Gamma_r = \mathbb{A}[y_1, \ldots, y_r]^{\mathfrak{S}_r}$. Let $s_\lambda \in \Gamma_r$ be the Schur polynomial associated to $\lambda \in \Pi_r$.

Fix some $n \in \mathbb{N}$ and let $\mathbf{i} \in \mathcal{A}_r^n$. From $s_\lambda(X_1^{-1}, \dots, X_r^{-1}) \in Z(\widehat{\mathbf{H}}_r)$ it follows that $e_{\mathbf{i}} s_\lambda(X_1^{-1}, \dots, X_r^{-1}) \in \widehat{\mathbf{H}}_{\mathbf{ii}}$. Define polynomials $J^{\mathbf{i}}_{\lambda,\sigma} \in \mathbb{Z}[v, v^{-1}]$ by the relation

$$e_{\mathbf{i}} s_\lambda(X_1^{-1}, \dots, X_r^{-1}) = (-v)^{(n-1)|\lambda|} \sum_{\sigma \in \mathfrak{S}_{\mathbf{i}} \backslash \widehat{\mathfrak{S}}_r / \mathfrak{S}_{\mathbf{i}}} J^{\mathbf{i}}_{\lambda,\sigma} \mathbf{c}_\sigma.$$

In this section we give an explicit expression for $J^{\mathbf{i}}_{\lambda,\sigma}$ involving (parabolic) affine Kazhdan-Lusztig polynomials of type A.

Remark. It is clear that (up to a power of v) $J^{\mathbf{i}}_{\lambda,\sigma}$ depends only on $\mathfrak{S}_{\mathbf{i}}$ rather than on $\mathbf{i}$. In particular, any parabolic subgroup $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_t}$ occurs as $\mathfrak{S}_{\mathbf{i}}$ for some $\mathbf{i} \in \mathcal{A}_r^n$ as soon as $n \geq t$.

2.2 We first make some preliminary definitions. We will represent a partition λ by its associated Young diagram in the usual fashion. We will consider diagrams where the (i, j)-box has content $i - j + r_0$ *mod* n for some fixed $r_0 \in \mathbb{Z}/n\mathbb{Z}$ and call the resulting tableau the *partition* λ *with residue* r_0. We will say that a box with content $j \in \mathbb{Z}/n\mathbb{Z}$ can be *added* to the partition λ with residue r_0 if there exists a partition λ' with residue r_0 such that λ'/λ is a single box with content j. For example, when $n = 3$, the partition $\lambda = (421)$ with residue 1 is

1				
2	3			
3	1	2		
1	2	3	1	2

and the dotted lines correspond to addable boxes.

To each $\mathbf{p} \in (\mathbb{Z}^+)^r$ and $\mathbf{i} \in \mathcal{A}_r^n$ we associate a multipartition (with residues) $\mathcal{M}_{\mathbf{i}}(\mathbf{p})$. First, we attach a diagram (not a partition!)

$$D_{\mathbf{p}} = \{(i, j) \mid 0 < j \leq \mathbf{p}_i\} \subset \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}^+,$$

where we fill the (i, j)-box with the content $\mathbf{i}_i + \mathbf{p}_i - j$ *mod* n.

Example 1. Suppose $r = n = 5$, $\mathbf{i} = (1, 2, 3, 4, 5)$ and $\mathbf{p} = (4, 3, 4, 3, 5)$. Then $D_{\mathbf{p}}$ is

				5
1		3		1
2	2	4	4	2
3	3	5	5	3
4	4	1	1	4

Now consider the horizontal slices $s_k = D_{\mathbf{p}} \cap (\mathbb{Z}/r\mathbb{Z} \times \{k\})$ and let k_0 be maximal such that $s_{k_0} \neq \emptyset$. We construct the multipartition with residues $\mathcal{M}_{\mathbf{i}}(\mathbf{p})$ by successively adding the boxes from $s_{k_0}, \dots, s_1$ in the following way. Set $\mathcal{M}^{k_0+1} = \emptyset$. Suppose $\mathcal{M}^i = (\lambda_i^{(1)}, \dots, \lambda_i^{(t)})$ is known. Then $\mathcal{M}^{i-1} = (\lambda_{i-1}^{(1)}, \dots, \lambda_{i-1}^{(r)})$ is obtained from $\mathcal{M}^i$ by adding the boxes from s_i (possibly creating new partitions) in such a way that

i) For every $1 \leq v \leq r$, $\lambda_{i-1}^{(v)}/\lambda_i^{(v)}$ is a skew tableau with *at most* one box in each row,

ii) $\mathcal{M}^{i-1}$ is maximal for the following order :
$(\lambda_{i-1}^{(1)}, \lambda_{i-1}^{(2)}, \dots) \geq (\mu_{i-1}^{(1)}, \mu_{i-1}^{(2)}, \dots)$ *if there exists w such that*

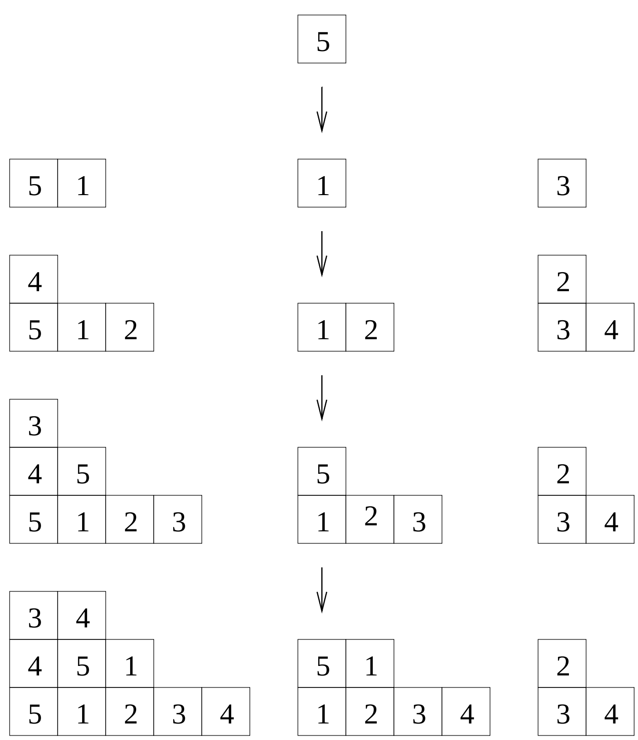
$$\lambda_{i-1}^{(l)} = \mu_{i-1}^{(l)} \;\; for\ 1 \leq l < w \qquad and \qquad \lambda_{i-1}^{(w)} \geq \mu_{i-1}^{(w)},$$

where $\geq$ stands for the usual dominance order of partitions,

iii) If several new partitions appear in $\mathcal{M}^{i-1}$ then they are in increasing order of their residue.

Set $\mathcal{M}_{\mathbf{i}}(\mathbf{p}) = \mathcal{M}^1$. We note that condition iii) above is not essential for the rest of the paper and here only to fix notations.

Examples. i) Let $r = n$ and $\mathbf{i} = (1, \dots, r)$. Suppose that $\mathbf{p}$ is *antidominant* up to cyclic permutation, i.e there exists $i \in \mathbb{Z}/r\mathbb{Z}$ such that $\mathbf{p}_i \geq \mathbf{p}_{i-1} \geq \dots \geq \mathbf{p}_{i+1}$. Let λ be the associated partition. Then $\mathcal{M}_{\mathbf{i}}(\mathbf{p})$ consists of the single partition λ with residue i.
ii) Consider r, n, $\mathbf{i}$ and $\mathbf{p}$ as in example 1. Then the algorithm for computing $\mathcal{M}_{\mathbf{i}}(\mathbf{p})$ runs as follows :



For $\sigma \in \mathfrak{S}_{\mathbf{i}}\backslash\widehat{\mathfrak{S}}_r/\mathfrak{S}_{\mathbf{i}}$ we let σ_0 be the longest element in σ and we set $\mathbf{i}\bullet\sigma = \mathbf{i}\cdot\sigma_0 - \mathbf{i} \in \mathbb{Z}^r$. For each σ such that $\mathbf{i}\bullet\sigma \in (\mathbb{Z}^+)^r$ we set $\mathcal{M}(\sigma) = \mathcal{M}_{\mathbf{i}}(\mathbf{i}\bullet\sigma)$. Write $\mathcal{M}(\sigma) = (\sigma^{(1)}, \dots, \sigma^{(l)})$ where $\sigma^{(l)} \neq \emptyset$, and $\mathbf{r}_\sigma = (r_1, \dots, r_l)$ where $r_i \in \mathbb{Z}/n\mathbb{Z}$ is the residue of $\sigma^{(i)}$.

2.3 Let $l \in \mathbb{N}$. Let $(\sigma^{(1)}, \dots, \sigma^{(l)})$, $(\mu^{(1)}, \dots, \mu^{(l)})$ be any l-multipartitions and let $\mathbf{r} = (r_1, \dots, r_l) \in (\mathbb{Z}/n\mathbb{Z})^l$. Choose some $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$ such that $s_i \equiv r_i \pmod{n}$. For $i = 1, \dots l$ and $j \in \mathbb{N}$ we set $u_j^{(i)} = s_i + \sigma_j^{(i)} + 1 - j$ and $v_j^{(i)} = s_i + \mu_j^{(i)} + 1 - j$. Consider, for $t \gg 0$

$$\mathbf{u} = (u_1^{(1)}, \dots, u_t^{(1)}, u_1^{(2)}, \dots, u_t^{(2)}, \dots, u_t^{(l)}),$$

$$\mathbf{v} = (v_1^{(1)}, \dots, v_t^{(1)}, v_1^{(2)}, \dots, v_t^{(2)}, \dots, v_t^{(l)}).$$

Finally, we put

$$\mathbf{P}^{-,\mathbf{s}}_{(\mu^{(1)},\dots,\mu^{(l)}),(\sigma^{(1)},\dots,\sigma^{(l)})} = P^-_{\mathbf{v},\mathbf{u}}.$$

Now let $\mathbf{s}$ be in the asymptotic range $s_1 \gg s_2 \gg \cdots \gg s_l$ and set

$$\mathbf{P}^{-,\mathbf{r}}_{(\mu^{(1)},\dots,\mu^{(l)}),(\sigma^{(1)},\dots,\sigma^{(l)})} = P^-_{\mathbf{v},\mathbf{u}}.$$

This polynomial is independent of the choices of $\mathbf{s}$ and t in the given asymptotic range (this follows for instance from [U] Section 4 and [S], Theorem 4.1). These can be thought of as some "stabilization" of polynomials $P^-_{\mu+\rho_r,\sigma+\rho_r}$ of type $\tilde{A}_r$ as r tends to infinity (see [LT]). Moreover, it is easy to see that when $l = 1$, $\mathbf{P}^{-,\mathbf{r}}$ is independent of $\mathbf{r}$ and we will omit it.

2.4 For any multipartition $\mu = (\mu^{(1)}, \dots, \mu^{(l)})$ and $\mathbf{r} = (r_1, \dots, r_l) \in (\mathbb{Z}/n\mathbb{Z})^l$ we set $\mu' = ((\mu^{(l)})', \dots, (\mu^{(1)})')$ and $\mathbf{r}' = (-r_l, \dots, -r_1)$.

2.5 The following is the main result of this paper, and will be proved in Section 5.

Theorem. *We have*

$$e_{\mathbf{i}} s_\lambda(X_1^{-1}, \dots, X_r^{-1}) = (-v)^{(n-1)|\lambda|} \sum_{\sigma\ |\mathbf{i}\bullet\sigma \in (\mathbb{Z}^+)^r} J^{\mathbf{i}}_{\lambda,\sigma} \mathbf{c}_\sigma$$

where

$$J^{\mathbf{i}}_{\lambda,\sigma} = \sum_{\substack{\nu_1,\dots,\nu_l \\ \mu_1,\dots,\mu_l}} c^\lambda_{\mu_1,\dots,\mu_l} v^{\sum (b-1)|\mu_b|} \mathbf{P}^-_{\nu_1, n\mu_1'} \cdots \mathbf{P}^-_{\nu_l, n\mu_l'} \mathbf{P}^{-,\mathbf{r}_\sigma'}_{\nu,\mathcal{M}(\sigma)'}$$

and $\nu = (\nu_1, \dots, \nu_l)$. *Here* $c^\lambda_{\mu_1,\dots,\mu_l}$ *is the (generalized) Littlewood-Richardson coefficient.*

Examples. i) Suppose that $n = 1$. Then $\mathbf{i} = (1^r)$ and $\mathfrak{S}_{\mathbf{i}} = \mathfrak{S}_r$. Moreover, $\mathfrak{S}_r \backslash \widehat{\mathfrak{S}}_r / \mathfrak{S}_r = \Pi_r$ and for $\sigma \in \Pi_r$ we have $\mathcal{M}(\sigma) = \sigma$ and $l = 1$. Hence the above theorem reduces to $J^{\mathbf{i}}_{\lambda,\sigma} = \sum_\nu \mathbf{P}^-_{\nu,\lambda'} \mathbf{P}^-_{\nu,\sigma'} = \delta_{\lambda,\sigma}$, i.e

$$\Big(\sum_{w \in \mathfrak{S}_r} T_w\Big) s_\lambda(X_1^{-1}, \dots, X_r^{-1}) = \mathbf{c}_\lambda,$$

in accordance with [L1].
ii) Let $r = n$ and $\mathbf{i} = \rho$ (i.e $\mathfrak{S}_{\mathbf{i}} = \{1\}$). Let $\lambda = (1^l)$, $l \leq r$ be a minuscule weight. Then in the above expression for $J^{\mathbf{i}}_{\lambda,\sigma}$ the only nonzero terms correspond to the case when μ_i is also minuscule for all i. We obtain an expression for $s_\lambda(X_1^{-1}, \dots, X_r^{-1})$ analogous to Theorem 1.1 in [H2] for $G = GL(r)$ (but which involves Kazhdan-Lusztig polynomials rather than R-polynomials). Note that [H1], Proposition 5 also easily follows from the above theorem.

3 Hall algebra of a cyclic quiver

3.0 Notations. In this section we fix a positive integer n. Let (ϵ_i), $i \in \mathbb{Z}/n\mathbb{Z}$ be the canonical basis of $\mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$. For $i \in \mathbb{Z}/n\mathbb{Z}$ and $l \in \mathbb{N}^*$, define the *cyclic segment* $[i; l)$ to be the image of the projection to $\mathbb{Z}/n\mathbb{Z}$ of the segment $[i_0, i_0 + l - 1] \subset \mathbb{Z}$ for any $i_0 \equiv i \pmod n$. A *cyclic multisegment* is a linear combination $\mathbf{m} = \sum_{i,l} a_i^l [i; l)$ of cyclic segments with coefficients $a_i^l \in \mathbb{N}$. Let $\mathcal{M}$ be the set of cyclic multisegments. For $\mathbf{m} \in \mathcal{M}$ we set $\dim \mathbf{m} = \sum a_i^l (\epsilon_i + \cdots + \epsilon_{i+l-1})$. Note that $\mathcal{M}$ is canonically isomorphic to Π^n : to $\mathbf{m} = \sum a_i^l [i; l)$ we associate the multipartition $(\lambda^{(1)}, \dots, \lambda^{(n)})$ with $\lambda^{(i)} = (1^{a_i^1} 2^{a_i^2} \dots)$.

3.1 Let Q be the quiver of type $\tilde{A}_{n-1}$, i.e the oriented graph with vertex set $I = \mathbb{Z}/n\mathbb{Z}$ and edge set $\Omega = \{(i, i+1), i \in I\}$. For any I-graded $\mathbb{F}$-vector space $V = \bigoplus_{i \in I} V_i$, let $E_V \subset \bigoplus_{(i,j) \in \Omega} \mathrm{Hom}\ (V_i, V_j)$ denote the space of nilpotent representations of Q. The group $G_V = \prod_{i \in I} GL(V_i)$ acts on E_V by conjugation. For each $i \in I$ there exists a unique simple Q-module S_i of dimension ϵ_i, and for each pair $(i, l) \in I \times \mathbb{N}^*$ there exists a unique (up to isomorphism) indecomposable Q-module $S_{i;l}$ of length l and tail S_i. Furthermore, every nilpotent Q-module M admits an essentially unique decomposition

$$M \simeq \bigoplus_{i,l} a_i^l S_{i;l}. \tag{3.1}$$

We denote by $\overline{\mathbf{m}}$ the isomorphism class of Q-modules corresponding (by (3.1)) to the multisegment $\mathbf{m} = \sum_{i,l} a_i^l [i; l)$. For $\mathbf{m} \in \mathcal{M}$ with $\dim \mathbf{m} = \mathbf{d}$ and $V_{\mathbf{d}}$ an I-graded vector space of dimension $\mathbf{d}$, we let $O_{\mathbf{m}} \subset E_{V_{\mathbf{d}}}$ be the $G_{V_{\mathbf{d}}}$-orbit consisting of representations in the class $\overline{\mathbf{m}}$, and we let $\mathbf{1}_{\mathbf{m}} \in \mathbb{C}_G(V_{\mathbf{d}})$ be the characteristic function of $O_{\mathbf{m}}$. Finally, we set $\mathbf{f}_{\mathbf{m}} = q^{-\dim O_{\mathbf{m}}} \mathbf{1}_{\mathbf{m}}$. We will write $\mathbf{m} < \mathbf{n}$ if $\mathcal{O}_{\mathbf{m}} \subset \overline{\mathcal{O}_{\mathbf{n}}}$.

3.2 Set $\mathbf{U}_n^- = \bigoplus_{\mathbf{d}} \mathbb{C}_G(E_{V_{\mathbf{d}}})$. Note that, by definition, $(\mathbf{f}_{\mathbf{m}})_{\mathbf{m} \in \mathcal{M}}$ is a $\mathbb{C}$-basis of $\mathbf{U}_n^-$. The space $\mathbf{U}_n^-$ is endowed with the structure of a (Hall) algebra (see [L1]). We use the definitions of [VV], [S]. Moreover, the structure constants for this algebra are polynomials in q, and one can consider $\mathbf{U}_n^-$ as an $\mathbb{A}$-algebra with

$q = v^{-1}$. The algebra $\mathbf{U}_n^-$ is naturally $\mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$-graded and we denote by $\mathbf{U}_n^-[\mathbf{d}]$ the component of degree $\mathbf{d}$. Let $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ denote the Lusztig integral form of the quantum affine algebra of type $\tilde{A}_{n-1}$ and let $e_i^{(l)}, k_i, f_i^{(l)}$, $i \in I$, $l \in \mathbb{N}$ be the divided powers of the standard Chevalley generators. Let $\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ generated by $f_i^{(l)}, i \in I, l \in \mathbb{N}^*$. It is known that the map $f_i^{(l)} \mapsto \mathbf{f}_{l\epsilon_i}$ extends to an embedding of the algebras $\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) \hookrightarrow \mathbf{U}_n^-$.

3.3 For $\mathbf{m} \in \mathcal{M}$, set

$$\mathbf{b_m} = \sum_{i,\mathbf{n}} v^{-i+\dim O_\mathbf{m} - \dim O_\mathbf{n}} \dim \mathcal{H}^i_{O_\mathbf{n}}(IC_{O_\mathbf{m}})\mathbf{f_n}, \tag{3.2}$$

where $\mathcal{H}^i_{O_\mathbf{n}}(IC_{O_\mathbf{m}})$ is the stalk over a point of $O_\mathbf{n}$ of the ith intersection cohomology sheaf of the closure $\overline{O}_\mathbf{m}$ of $O_\mathbf{m}$. Then $\mathbf{B} = \{\mathbf{b_m}\}$ is the canonical basis of $\mathbf{U}_n^-$, introduced in [VV].

3.4 Let $L, L' \in Y$ be two n-step periodic flags in $\mathbb{L}^r$ satisfying $L' \subset L$. Following Lusztig (see [L2],[GV]) we associate to such a pair a nilpotent representation of $\tilde{A}_{n-1}$ of graded dimension $(\dim_\mathbb{F}(L_i/L'_i))_{\bar{i}\in\mathbb{Z}/n\mathbb{Z}}$. Let us denote by L/L' this $\tilde{A}_{n-1}$-module. Set

$$a(L', L) = \sum_{i=1}^n \dim_\mathbb{F}(L_i/L'_i)(\dim_\mathbb{F}(L'_{i+1}/L'_i) - \dim_\mathbb{F}(L_i/L'_i)).$$

Define a map $\Theta :\ \mathbf{U}_n^- \to \widehat{\mathbf{S}}_{n,r}$ by

$$\Theta(f)(L', L) = q^{-a(L',L)} f(L/L') \quad \text{if } L' \subseteq L$$

and $\Theta(f)(L, L') = 0$ if $L' \nsubseteq L$.

In order to describe Θ, we consider the following parametrization of the collection of $\mathbb{G}$-orbits in $Y \times Y$. Let $M_{r,n}$ be the set of $\mathbb{Z}\times\mathbb{Z}$-matrices $\mathbf{s} = (s_{ij})_{i,j\in\mathbb{Z}}$ with entries in $\mathbb{N}$ such that $s_{i+n,j+n} = s_{i,j}$ and $\sum_j \sum_{i=1}^n s_{ij} = r$. To each such $\mathbf{s} \in M_{r,n}$ we associate the $\mathbb{G}$-orbit $Y_\mathbf{s}$ whose elements are the pairs (L, L') for which

$$s_{ij} = \dim_\mathbb{F}\left(\frac{L_i \cap L'_j}{(L_i \cap L'_{j-1}) + (L_{i-1} \cap L'_j)}\right).$$

For $\mathbf{i}, \mathbf{j} \in \mathcal{A}_r^n$ we denote by $M_\mathbf{ij}$ the set of all $\mathbf{s}$ such that $Y_\mathbf{s} \subset Y_\mathbf{i} \times Y_\mathbf{j}$. It is easy to see that

$$M_{ij} = \{\mathbf{s} \in M_{r,n} \mid \sum_j s_{ij} = \#\mathbf{i}^{-1}(i),\ \sum_i s_{ij} = \#\mathbf{j}^{-1}(j)\}.$$

In particular, $M_\mathbf{ij}$ is naturally identified with $\mathfrak{S}_\mathbf{i}\backslash\widehat{\mathfrak{S}}_r/\mathfrak{S}_\mathbf{j}$.

Let us associate to each $\mathbf{m} = \sum a_i^l[i;l)$ the matrix $(m_{i,j}) \in \bigcup_r M_{r,n}$ with $m_{i,j} = a_i^{j-i+1}$. The set

$$M^+ = \{(m_{i,j})_{i,j\in\mathbb{Z}} \mid m_{i+n,j+n} = m_{i,j},\ i > j \Rightarrow m_{i,j} = 0\}$$

is then identified with $\mathcal{M}$. If $\mathbf{i} \in \mathcal{A}_r^n$ and $\mathbf{m} \in M^+$ we let $\mathbf{m}^{\mathbf{i}} \in \bigcup_{\mathbf{j}} M_{\mathbf{ij}}$ be the matrix whose (i,j)th entry is

$$\delta_{ij}(\#\mathbf{i}^{-1}(j+1) - \sum_{k\le j} m_{kj}) + (1-\delta_{ij})m_{i+1,j}.$$

Proposition ([VV]). *The map* $\Theta : \mathbf{U}_n^- \to \widehat{\mathbf{S}}_{n,r}$ *is an algebra morphism satisfying* $\Theta(\overline{u}) = \tau(\Theta(u))$ *for every* $u \in \mathbf{U}_n^-$. *Furthermore,*

$$\Theta(\mathbf{f}_{\mathbf{m}}) = \sum_{\mathbf{i}\,\mid\,\mathbf{m}^{\mathbf{i}}\in M^+} \tilde{T}_{\mathbf{m}^{\mathbf{i}}}, \qquad \Theta(\mathbf{b}_{\mathbf{m}}) = \sum_{\mathbf{i}\,\mid\,\mathbf{m}^{\mathbf{i}}\in M^+} \mathbf{c}_{\mathbf{m}^{\mathbf{i}}}.$$

It follows from the above Proposition that $\mathbf{T}_{n,r}$ is endowed with a canonical $\mathbf{U}_n^-$-module structure.

3.5 Let e_i', $i \in \mathbb{Z}/n\mathbb{Z}$ be the adjoint of the left multiplication by $\mathbf{f}_i$. Set $\mathbf{R} = \bigcap_i \operatorname{Ker} e_i' \subset \mathbf{U}_n^-$. Let us identify the ring of symmetric polynomials Γ_r with Z_r^- by $y_i \mapsto X_i^{-1}$.

Theorem ([S]). *The vector space* $\mathbf{R}$ *is a graded central subalgebra of* $\mathbf{U}_n^-$ *and the multiplication map induces an isomorphism* $\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) \otimes_{\mathbb{A}} \mathbf{R} \xrightarrow{\sim} \mathbf{U}_n^-$. *Moreover there exists surjective algebra morphisms* $i_r : \mathbf{R} \to Z_r^-$ *and an algebra isomorphism* $i : \mathbf{R} \to \Gamma$ *such that*

$$\rho_r \circ \Theta = \sigma_r \circ i_r, \qquad i = \lim_{\longleftarrow} i_r.$$

Let $s_\lambda \in \Gamma$ be the Schur polynomial associated to $\lambda \in \Pi$, and set $a_\lambda = i^{-1}(s_\lambda)$. Then $i_r(a_\lambda) = s_\lambda(X_1^{-1}, \dots, X_r^{-1})$ for any $r \geq l(\lambda)$. For $\mathbf{m} \in \mathcal{M}$, define polynomials $J_{\mathbf{m}}^\lambda \in \mathbb{Z}[v, v^{-1}]$ by

$$a_\lambda = \sum_{\mathbf{m}} J_{\lambda,\mathbf{m}} \mathbf{b}_{\mathbf{m}}. \tag{3.3}$$

Corollary. *For any* $r \in \mathbb{N}$ *and* $\mathbf{i} \in \mathcal{A}_r^n$ *we have*

$$\mathbf{e}_{\mathbf{i}} s_\lambda(X_1^{-1}, \dots X_r^{-1}) = \sum_{\mathbf{m}\,\mid \mathbf{m}^{\mathbf{i}}\in M^+} J_{\lambda,\mathbf{m}} \mathbf{c}_{\mathbf{m}^{\mathbf{i}}}. \tag{3.4}$$

Proof. This follows by applying a_λ to $\mathbf{e}_{\mathbf{i}} \cdot 1 \in \mathbf{T}_{n,r}$, and using Theorem 3.5 and Proposition 3.4. ☐

Remarks. i) Let us consider the case $n = 1$ and $\mathbf{i} = (1^r)$. Then $\mathcal{M} = \Pi$ and $\mathbf{U}_1^- = \mathbf{R} \overset{i}{\simeq} \Gamma$, and it is known that i identifies the Poincaré-Birkhoff-Witt basis element $\mathbf{f}_\lambda$ with the Hall-Littlewood polynomial P_λ (see [Mac], Chap. III). In

particular, $K^\lambda_\mu(v)$ is the Kostka-Foulkes polynomial and from (3.4) we recover the well-known result of Lusztig ([L1]) concerning the Satake isomorphism

$$(\sum_{\sigma\in\mathfrak{S}_r} T_\sigma)s_\lambda(X_1^{-1},\dots,X_r^{-1}) = \sum_{\mu\in\Pi} K^\lambda_\mu(v)\tilde{T}_{\mathfrak{S}_r\mu\mathfrak{S}_r}.$$

ii)Define the following symmetric bilinear form on $\mathbf{U}_n^-$ (the *Green's scalar product*) :

$$\langle \mathbf{f}_\mathbf{m}, \mathbf{f}_{\mathbf{m}'}\rangle = v^{-2\dim\mathrm{Aut}(\mathbf{m})}\frac{(1-v^2)^{|\mathbf{m}|}}{|\mathrm{Aut}(\mathbf{m})|}\delta_{\mathbf{m},\mathbf{m}'},$$

where $\mathrm{Aut}(\mathbf{m})$ stands for the group of automorphism of any representation in the orbit $O_\mathbf{m}$ and $|\sum a^l_i[i;l)| = \sum_{i,l} la^l_i$. It is natural to consider the restriction of this scalar product $(\ ,\ )$ on $\mathbf{U}_n^-$ to $\mathbf{R} \overset{i}{\simeq} \Gamma$. Let $\mathcal{M}^{\mathrm{per}}$ denote the set of multisegments of the form $\mathbf{m} = \sum a^l_i[i;l)$ such that $a^l_i = a^l_j$ for all i,j. By [S], Proposition 2.4 we have

$$\mathbf{R} = \big(\bigoplus_{\mathbf{m}\notin\mathcal{M}^{\mathrm{per}}} \mathbb{A}\mathbf{b}_\mathbf{m}\big)^\perp.$$

Hence the restriction of $(\ ,\ )$ to $\mathbf{R}$ is nondegenerate. When $n=1$ this restriction coincides, up to a constant, with the Hall-Littlewood scalar product. Let $(p_\mu)_{\mu\in\Pi}$ be the basis of power-sum symmetric functions and let $z_{(1^{m_1}2^{m_2}\dots)} = \prod_i m_i! i^{m_i}$.

Conjecture. *The restriction of Green's scalar product on $\mathbf{R}\subset\mathbf{U}_n^-$ is given by*

$$(p_\lambda, p_\mu) = \delta_{\lambda,\mu} z_\lambda v^{-2(n-1)|\lambda|}(1-v^2)^{n|\lambda|}\prod_{i=1}^{l(\lambda)}\frac{1-v^{-2n\lambda_i}}{(1-v^{-2\lambda_i})^2}.$$

This scalar product can be seen as a higher-rank analogue of the Hall-Littlewood scalar product.

4 Uglov's Fock spaces

4.1 Let n,l be positive integers and let $\mathbf{s}_l\in\mathbb{Z}^l$. Following [JMMO], Uglov attached to this data an integrable $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-module $\Lambda^\infty_{\mathbf{s}_l}$ equipped with a distinguished $\mathbb{A}$-basis $\{|\lambda_l,\mathbf{s}_l\rangle\}$, $\lambda_l\in\Pi^l$ (the *higher-level Fock space*, see [U], Section 1). The Fock space $\Lambda^\infty_{\mathbf{s}_l}$ is also endowed with an action of a Heisenberg algebra $\mathcal{H}$ generated by operators B_m, $m\in\mathbb{Z}^*$ (see [U], Sections 4.2, 4.3). Moreover, the $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$-action and the $\mathcal{H}$-action commute.

Remark. When $l=1$, Uglov's Fock space coincides with the Fock space Λ^∞ introduced in [KMS].

4.2 We now extend the action of $\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)$ on $\Lambda^\infty_{\mathbf{s}_l}$ to an action of $\mathbf{U}_n^-$. We follow the method of Varagnolo-Vasserot [VV], Section 5. Let $\mathbf{U}_\infty^-$ be the Hall algebra of the quiver of type A_∞. It is known that $\mathbf{U}_\infty^- = \mathbf{U}_v^-(\mathfrak{sl}_\infty)$. Let f_i, $i\in\mathbb{Z}$ be the standard generator corresponding to the vertex i.

We associate to each $\lambda_l = (\lambda^{(1)}, \dots \lambda^{(l)}) \in \Pi^l$ an l-tuple of Young tableaux $(T_1, \dots, T_l)$ such that

i) T_d is of shape $\lambda^{(d)}$ for $d = 1, \dots, l$,

ii) The (i, j)-box of T_d is filled with content $s_d + i - j$.

If λ_l and μ_l are two l-multipartitions such that $\gamma = \mu_l \backslash \lambda_l$ corresponds to a box with content $k \in \mathbb{Z}$, we say that γ is an *addable k-box* of λ_l and a *removable k-box* of μ_l. Let γ, γ' be two addable k-boxes of λ_l. We say that $\gamma < \gamma'$ if γ and γ' belong to T_d and $T_{d'}$ respectively and $d < d'$.
Let $\lambda_l, \mu_l \in \Pi^l$ be such that $\mu_l \backslash \lambda_l$ is a k-box. Define

$$\begin{aligned} N^{>}(\mu_l, \lambda_l) = & \#\{addable\ k-boxes\ \gamma'\ of\ \lambda_l\ such\ that\ \gamma' > \gamma\} \\ & - \#\{removable\ k-boxes\ \gamma'\ of\ \lambda_l\ such\ that\ \gamma' > \gamma\}. \end{aligned}$$

Proposition. *The following endows $\Lambda^{\infty}_{\mathbf{s}_l}$ with a structure of a $\mathbf{U}^-_{\infty}$-module :*

$$f_k \cdot |\lambda_l, \mathbf{s}_l\rangle = \sum_{\mu_l} v^{N^{>}(\mu_l, \lambda_l)} |\mu_l, \mathbf{s}_l\rangle$$

where the sum ranges over all μ_l for which $\mu_l \backslash \lambda_l$ is a k-box.

Proof. Straightforward. □

Define operators $\mathbf{k}_k \in \mathrm{End}\,(\Lambda^{\infty}_{\mathbf{s}_l})$, $k \in \mathbb{Z}$ by $\mathbf{k}_k \cdot |\lambda_l, \mathbf{s}_l\rangle = v^{N_k(\lambda_l)} |\lambda_l, \mathbf{s}_l\rangle$ where

$$N_k(\lambda_l) = \#\{addable\ k-boxes\ of\ \lambda_l\} - \#\{removable\ k-boxes\ of\ \lambda_l\}.$$

Now let $d \in \mathbb{N}^{(\mathbb{Z})}$ and set $\overline{d} = (\overline{d}_1, \dots, \overline{d}_n)$ where $\overline{d}_i = \sum_{j \equiv i \pmod n} d_j$. Let V be a $\mathbb{Z}$-graded $\mathbb{F}$-vector space of dimension d and let $\overline{V}$ be the $\mathbb{Z}/n\mathbb{Z}$-graded $\mathbb{F}$-vector space with $\overline{V}_i = \bigoplus_{j \equiv i} V_j$. The collection of subspaces $\overline{V}_{\leq i} = \bigoplus_{j \leq i} V_j$ defines a filtration of $\overline{V}$ whose associated graded is V. Set

$$E_{\overline{V}, V} = \{x \in E_{\overline{V}} \mid x(\overline{V}_{\geq i}) \subset \overline{V}_{\geq i+1} \text{ for all } i\}.$$

Let $p : E_{\overline{V}, V} \to E_V$ be the projection onto the graded. Let $j : E_{\overline{V}, V} \subset E_V$ be the closed embedding. Following [VV], define a map $\gamma_d : \mathbf{U}^-_n[\overline{d}] \to \mathbf{U}^-_{\infty}[d]$ by

$$\begin{aligned} \gamma_{d|v=q^{-1}} :\ & \mathbb{C}_{G_{\overline{V}}}(E_{\overline{V}}) \to \mathbb{C}_{G_V}(E_V) \\ & f \mapsto q^{-h(d)} p_! j^*(f) \end{aligned}$$

where $h(d) = \sum_{i<j, i \equiv j} d_i(d_{j+1} - d_j)$.

For all $\lambda_l \in \Pi^l$ and $x \in \mathbf{U}^-_n$ we put

$$x \cdot |\lambda_l, \mathbf{s}_l\rangle = \sum_d \big(\gamma_d(x) \prod_{j<i, j \equiv i} \mathbf{k}_i^{d_j}\big) \cdot |\lambda_l, \mathbf{s}_l\rangle. \tag{4.1}$$

Then (see [VV] Section 6.2, and [A])

Proposition. *Formula (4.1) defines a representation $\Xi : \mathbf{U}^-_n \to \mathrm{End}\,(\Lambda^{\infty}_{\mathbf{s}_l})$ which extends Uglov's action of $\mathbf{U}^-_v(\widehat{\mathfrak{sl}}_n)$.*

Remarks. i) The number $h(d)$ has the following interpretation. Let $\mathcal{F}_d$ be the variety of filtrations of $\overline{V}$ whose associated graded is of dimension d. Then $\dim T^*\mathcal{F}_d = \dim G_{\overline{V}} + h(d)$.
ii) The map γ_d is "upper triangular" in the following sense. Let $x \in E_V$ and define $r(x) \in E_{\overline{V}}$ by $r(x)_i = \bigoplus_{j \equiv i} x_j$. Then $\gamma_d(\mathbf{f_m})(x) \neq 0 \Rightarrow r(x) \in \overline{\mathcal{O}_{\mathbf{m}}}$.

4.3 Let $\mathcal{H}^- \subset \mathcal{H}$ denote the subalgebra generated by B_{-m}, $m \in \mathbb{N}^*$. Define an algebra isomorphism $j : \Gamma \xrightarrow{\sim} \mathcal{H}^-$ by setting $j(p_m) = B_{-m}$, where p_m is the power-sum symmetric function. Recall the canonical map $i : \mathbf{R} \xrightarrow{\sim} \Gamma$ from Theorem 3.5.

Lemma. *We have* $\Xi_{|\mathbf{R}} = j \circ i$.

Proof (sketch). This is shown in a way similar to [VV]. We first consider the "limit" $\bigotimes^\infty$ of $\mathbf{T}_{n,r}$ when $r \to \infty$ (see [VV], Section 10). Then $\Lambda^\infty_{\mathbf{s}_l}$ is naturally embedded in a certain quotient of $\bigotimes^\infty$ (see [U], Section 3.3). In particular, the $\mathbf{U}^-_n$-action on $\mathbf{T}_{n,r}$ induces an action on $\bigotimes^\infty$ and on $\Lambda^\infty_{\mathbf{s}_l}$. Let Ξ' denote this last action. It follows from Theorem 3.5 and [U], Section 4 that $\Xi'_{|\mathbf{R}} = j \circ i$. Finally, an easy extension to the higher-level Fock space of the computation in [VV], Lemma 10.1 shows that $\Xi' = \Xi$. $\square$

5 Canonical bases of Fock spaces

5.1 We keep the settings of the previous Section. Uglov has defined a semilinear involution $a \mapsto \overline{a}$ on $\Lambda^\infty_{\mathbf{s}_l}$ ([U], Section 4.4) and two canonical bases $\{\mathbf{b}^\pm_{\lambda_l}\}_{\lambda_l \in \Pi^l}$ characterized by the following properties :

$$\overline{\mathbf{b}^\pm}_{\lambda_l} = \mathbf{b}^\pm_{\lambda_l},$$

$$\mathbf{b}^+_{\lambda_l} \in |\lambda_l\rangle + v \bigoplus_{\mu_l} \mathbb{S}|\mu_l\rangle, \qquad \mathbf{b}^-_{\lambda_l} \in |\lambda_l\rangle + v^{-1} \bigoplus_{\mu_l} \overline{\mathbb{S}}|\mu_l\rangle.$$

He furthermore computed the transition matrices $[\mathbf{b}^\pm_{\lambda_l} : |\mu_l, \mathbf{s}_l\rangle]$. In particular we have the following result.

Theorem ([U], 3.26).

$$\mathbf{b}^-_{\lambda_l} = \sum_{\mu_l} \mathbf{P}^{-,\mathbf{s}_l}_{\mu_l,\lambda_l} |\mu_l, \mathbf{s}_l\rangle.$$

Remark. When $l = 1$, Uglov's canonical bases coincide with the canonical bases considered by Leclerc-Thibon ([LT]). In that setting, the transition matrices above were first obtained by Varagnolo and Vasserot [VV].

5.2 Let us now consider the nondegenerate scalar product $(\ ,\ )$ on $\Lambda^\infty_{\mathbf{s}_l}$ for which $\{|\lambda_l, \mathbf{s}_l\rangle\}$ is orthonormal. Let $\{\mathbf{b}^{+*}_{\lambda_l}\}$ be the dual basis to $\{\mathbf{b}^+_{\lambda_l}\}$ with respect to the scalar product $(\ ,\ )$.

Define a semilinear isomorphism $\Lambda^\infty_{\mathbf{s}_l} \to \Lambda^\infty_{\mathbf{s}'_l}$, $u \mapsto u'$ by $|\lambda_l, \mathbf{s}_l\rangle' = |\lambda'_l, \mathbf{s}'_l\rangle$.

Proposition ([U], 5.14). *We have* $(\mathbf{b}^{+*}_{\lambda_l})' = \mathbf{b}^-_{\lambda'_l}$.

5.3 Let $\mathbf{B}_{\mathbf{s}_l} = \{\mathbf{b}^+_{\lambda_l}\}_{\lambda_l \in \Pi^l}$ be the (positive) canonical basis of $\Lambda^\infty_{\mathbf{s}_l}$.

Theorem. *Let $\mathbf{m} \in \mathcal{M}$. Then $\mathbf{b}_\mathbf{m} \cdot |0, \mathbf{s}_l\rangle \in \mathbf{B}_{\mathbf{s}_l} \cup \{0\}$.*

Proof. Lemma 4.3 implies that the $\mathbf{U}^-_n$-action on $\Lambda^\infty_{\mathbf{s}_l}$ is the same as that considered in [S], Section 4. The result follows from [S], Theorem 4.2. $\square$

5.4 Define a map $\tau_{\mathbf{s}_l} : \mathcal{M} \to \Pi^l \cup \{0\}$ by $\tau_{\mathbf{s}_l}(\mathbf{m}) = 0$ if $\mathbf{b}_\mathbf{m} \cdot |0, \mathbf{s}_l\rangle = 0$ and $\mathbf{b}_\mathbf{m} \cdot |0, \mathbf{s}_l\rangle = \mathbf{b}^+_{\tau_{\mathbf{s}_l}(\mathbf{m})}$ otherwise. This map is not easy to describe for a general $\mathbf{s}_l$. Nevertheless we have the following result.

Let $\mathbf{m} \in \mathcal{M}$ and let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be the associated n-multipartition. Let $r = \sum_i l(\lambda^{(i)})$. Set $\mathbf{i} = (1^{l(\lambda^{(1)})}, 2^{l(\lambda^{(2)})}, \dots) \in \mathcal{A}^n_r$ and

$$\mathbf{p} = (\lambda^{(1)}_1, \dots, \lambda^{(1)}_{l(\lambda^{(1)})}, \lambda^{(2)}_1, \dots) \in (\mathbb{Z}^+)^r.$$

Finally, let $\mathcal{M}_\mathbf{i}(\mathbf{p}) = (p^{(1)}, \dots, p^{(l)})$ and let $r_i \in \mathbb{Z}/n\mathbb{Z}$ be the residue of $p^{(i)}$.

Lemma. *Suppose that $s_1 \gg s_2 \gg \cdots \gg s_l$ and that $s_i \equiv r_i \pmod{n}$ for $i = 1, \dots l$. Then $\tau_{\mathbf{s}_l}(\mathbf{m}) = \mathcal{M}_\mathbf{i}(\mathbf{p})$.*

Proof. See appendix. $\square$

5.5 *Proof of Theorem 1.* Let $\sigma \in \mathfrak{S}_\mathbf{i} \backslash \widehat{\mathfrak{S}}_r / \mathfrak{S}_\mathbf{i}$. It follows from Corollary 3.5 that $J^\mathbf{i}_{\lambda,\sigma} = J_{\lambda,\mathbf{m}}$ if there exists $\mathbf{m} \in \mathcal{M}$ such that $\mathbf{m}^\mathbf{i} = \sigma$ and $J^\mathbf{i}_{\lambda,\sigma} = 0$ otherwise. From Section 3.4 we see that

$$\mathbf{m}^\mathbf{i} = \sigma \iff \mathbf{i} \bullet \sigma \in (\mathbb{Z}^+)^r \text{ and } \mathbf{m} = \sum_{j=1}^r [\mathbf{i}_j; (\mathbf{i} \bullet \sigma)_j).$$

Now we compute $J_{\lambda,\mathbf{m}}$. Let $l, \mathbf{p}, (r_i)_{i=1}^l$ be associated to $\mathbf{m}$ as in Section 5.4. Let $\mathbf{s}_l = (s_1, \dots, s_l)$ be in the asymptotic region $s_1 \gg s_2 \gg \cdots \gg s_l$ and satisfy $s_i \equiv r_i \pmod{n}$ for all i. We evaluate both sides of (3.3) on $|0, \mathbf{s}_l\rangle \in \Lambda^\infty_{\mathbf{s}_l}$. On the one hand, it follows from Lemma 4.3 and Uglov's description of the action of the Heisenberg algebra [U], Proposition 5.3 that

$$a_\lambda \cdot |0, \mathbf{s}_l\rangle = \sum_{\mu_1, \dots, \mu_l} c^\lambda_{\mu_1, \dots, \mu_l} v^{\sum (b-1)|\mu_b|} \left(\sum_{\nu_1, \dots, \nu_l} e_{\nu_1, \mu_1} \cdots e_{\nu_l, \mu_l} |(\nu_1, \dots, \nu_l), \mathbf{s}_l\rangle \right)$$

where $e_{\nu_i, \mu_i} \in \mathbb{Z}[v^{-1}]$ are defined by the relations $s_{\mu_i}|0\rangle = \sum_{\nu_i} e_{\nu_i, \mu_i} |\nu_i\rangle$ in the *level l=1* Fock space representation of $\mathbf{U}^-_n$. But by [LT], Theorem 6.9 we have $s_{\mu_i} \cdot |0\rangle = \mathbf{b}^-_{n\mu_i}$ and thus $e_{\nu_i, \mu_i} = \mathbf{P}^-_{\nu_i, n\mu_i}$. On the other hand, from Theorem 5.3 we have

$$\sum_\mathbf{n} J_{\lambda,\mathbf{n}} \mathbf{b}_\mathbf{n} \cdot |0, \mathbf{s}_l\rangle = \sum_{\mathbf{n}, \tau_{\mathbf{s}_l}(\mathbf{n}) \neq 0} J_{\lambda,\mathbf{n}} \mathbf{b}^+_{\tau_{\mathbf{s}_l}(\mathbf{n})}.$$

In particular, $J_{\lambda,\mathbf{m}} = \big((\mathbf{b}^+_{\tau_{\mathbf{s}_l}(\mathbf{m})})^*, a_\lambda |0, \mathbf{s}_l\rangle \big)$. But by Lemma 5.4 and Proposition 5.2,

$$(\mathbf{b}^+_{\tau_{\mathbf{s}_l}(\mathbf{m})})^* = (\mathbf{b}^+_{\mathcal{M}_\mathbf{i}(p)})^* = (\mathbf{b}^+_{\mathcal{M}(\sigma)})^* = (\mathbf{b}^-_{\mathcal{M}(\sigma)'})'.$$

Using the relations $(\overline{u}, v) = (u', \overline{v'})$ for any $u, v \in \Lambda^{\infty}_{\mathbf{s}_l}$ ([U], Proposition 5.13) and $\overline{a_\lambda \cdot |0, \mathbf{s}_l\rangle} = a_\lambda \cdot |0, \mathbf{s}_l\rangle$ ([U], Proposition 4.2) we get

$$J_{\lambda,\mathbf{m}} = (\mathbf{b}^-_{\mathcal{M}(\sigma)'}, a_\lambda \cdot |0, \mathbf{s}_l\rangle').$$

Now, from [LT], Theorem 7.13 i) we have $(\mathbf{b}^-_{n\mu})' = (-v)^{(n-1)|\mu|}\mathbf{b}^-_{n\mu'}$ in the level $l = 1$ Fock space. Thus

$$a_\lambda \cdot |0, \mathbf{s}_l\rangle =$$
$$=(-v)^{(n-1)|\lambda|} \sum_{\mu_1,\dots,\mu_l} c^\lambda_{\mu_1,\dots,\mu_l} v^{\sum_{b=1}^l (b-1)|\mu_b|} \Big(\sum_{\nu_1,\dots,\nu_l} \mathbf{P}^-_{\nu_1, n\mu'_1} \cdots \mathbf{P}^-_{\nu_l, n\mu'_l} |\nu, \mathbf{s}_l\rangle \Big)$$

where $\nu = (\nu_l, \dots, \nu_1)$. The theorem follows. $\square$

6 On the center of $\mathbf{U}^-_n$

In this section we give a simple geometric characterization of the central subalgebra $\mathbf{R} \subset \mathbf{U}^-_n$ in terms of the maps $\gamma_d : \mathbf{U}^-_n \to \mathbf{U}^-_\infty$ defined in Section 4.2.

6.1 Let $d \in \mathbb{N}^{(\mathbb{Z})}$ such that $d_i \in \{0, 1\}$ for all i. Then d is the dimension of a unique (noncyclic) multisegment $\mathbf{n}_d = \sum_{k=1}^t [i_k; l_k)$ in $\mathbb{Z}$ satisfying the following condition :

$$\forall\, j, k \qquad [i_k, l_k) \cup [i_j, l_j) \textit{ is not a segment.} \tag{6.1}$$

Let V_d be a $\mathbb{Z}$-graded $\mathbb{F}$-vector space of dimension d. Set $l(d) = \sum_k (l_k - 1)$. Note that it follows from (6.1) that E_{V_d} has a unique open G_{V_d}-orbit, say $\mathcal{O}_d$.

Lemma. *Suppose that $i_1 \gg i_2 \gg \cdots \gg i_t$ and set $\mathbf{i}_t = (i_1, \dots, i_t)$. Then for any $\mathbf{f} \in \mathbf{U}^-_\infty[d]$ we have*

$$\mathbf{f} \cdot |0, \mathbf{i}_t\rangle = v^{-l(d)} \mathbf{f}_{|\mathcal{O}_d} |((l_1), \dots, (l_k)), \mathbf{i}_t\rangle.$$

Proof. Note that $E_{V_d} = \prod_{k=1}^t E_{V_d(k)}$ where $V_d(k) = \bigoplus_{l=0}^{l_k-1} \mathbb{F}V_{i_k+l}$. Let $f_k \in E_{V_d(k)}$ for $k = 1, \dots, t$. From (6.1) and Section 4.2 we deduce that

$$f_1 \cdots f_t \cdot |0, \mathbf{i}_t\rangle = \sum_{\nu_1,\dots,\nu_t} d_1(\nu_1) \cdots d_t(\nu_t) |(\nu_1, \dots, \nu_t), \mathbf{i}_t\rangle$$

where $f_k|0, i_k\rangle = \sum_\nu d_k(\nu)|\nu, \mathbf{i}_k\rangle$ in the level $l = 1$ Fock space. But from [VV], Proposition 5., it is easy to see that $f_k \cdot |0, \mathbf{i}_k\rangle = v^{-(l_k-1)}(f_k)_{|\mathcal{O}_d(k)}|(l_k), i_k\rangle$ where $\mathcal{O}_d(k) \subset E_{V_d(k)}$ is the open orbit. $\square$

6.2 Recall the element $a_\lambda = i^{-1}(s_\lambda) \in \mathbf{R}$. For any $\lambda, \mu \in \Pi$ let $K^\lambda_\mu \in \mathbb{N}$ be the Kostka number.

Theorem. *Let $d \in \mathbb{N}^{(\mathbb{Z})}$ such that $d_i \in \{0, 1\}$. Then*

$$\gamma_d(a_\lambda)_{|\mathcal{O}_d} = v^{l(d)+h(d)} K^\lambda_{(u_1,\dots,u_t)}$$

if there exists $i_k, u_k \in \mathbb{Z}$, $k = 1, \dots, t$ such that $\mathbf{n}_d = \sum_{k=1}^t [i_k; nu_k)$, and $\gamma_d(a_\lambda)_{|\mathcal{O}_d} = 0$ otherwise.

Proof. Without loss of generality we may assume that $\mathbf{n} = \sum_{k=1}^{t} [i_k; l_k)$ where $i_1 > i_2 > \cdots > i_t$. Choose $d' = \cup_{k=1}^{t} [i'_k; l_k)$ where $i'_k \equiv i_k \pmod{n}$ and $i'_1 \gg i'_2 \gg \ldots \gg i'_t$. Let $\xi : E_{V'_d} \xrightarrow{\sim} E_{V_d}$ be the obvious isomorphism. Then $\xi \circ \gamma_{d'} = \gamma_d$. Now let us consider the Fock space $\Lambda^{\infty}_{\mathbf{i}'_t}$ where $\mathbf{i}'_t = (i'_1, \ldots, i'_t)$. Using [U], Proposition 5.3 we have

$$
\begin{aligned}
&\big(a_\lambda \cdot |0, \mathbf{i}'_t\rangle, |((l_1), \ldots, (l_t)), \mathbf{i}'_t\rangle\big) \\
&\qquad\qquad = \sum_{\mu_1, \ldots, \mu_t} c^{\lambda}_{\mu_1, \ldots, \mu_t} v^{\sum_b (b-1)|\mu_b|} \mathbf{P}^{-}_{(l_1), n\mu_1} \cdots \mathbf{P}^{-}_{(l_t), n\mu_t} \\
&\qquad\qquad = \sum_{\mu_1, \ldots, \mu_t} \delta_{(l_1)=n\mu_1} \cdots \delta_{(l_t)=n\mu_t} c^{\lambda}_{\mu_1, \ldots, \mu_t} v^{\sum_b (b-1)|\mu_b|}
\end{aligned}
$$

Note that for any $u_1, \ldots, u_t \in \mathbb{Z}$ we have $c^{\lambda}_{(u_1), \ldots, (u_t)} = K^{\lambda}_{\mu}$ where $\mu \in \Pi$ is the partition with parts $\{u_1, \ldots, u_t\}$.

On the other hand, by Lemma 6.1

$$
\big(a_\lambda \cdot |0, \mathbf{i}'_t\rangle, |((l_1), \ldots, (l_t)), \mathbf{i}'_t\rangle\big) = v^{\epsilon(d', \mathbf{i}'_t) - l(d)} \gamma_{d'}(a_\lambda)_{|\mathcal{O}_{d'}}
$$

where

$$
\epsilon(d', \mathbf{i}'_t) = \sum_{l=1}^{t} \sum_{j \equiv i'_l; j < i'_l} d'_j.
$$

The result now follows from the easily checked identity

$$
\epsilon(d', \mathbf{i}'_t) = \sum_b (b-1)|\mu_b| + h(d')
$$

when there exists $u_k \in \mathbb{N}$, $k = 1, \ldots, t$ such that $d' = \cup_k [i'_k, nu_k)$ and $\mu_k = (u_k)$. $\square$

Remark. It follows from Remark 4.2 ii) that the previous theorem gives a characterization of the central element a_λ.

7 Appendix

In this appendix we prove Lemma 5.4.

A.1 As in [U], Section 4, define a partial order on Π^l (depending on $\mathbf{s}_l$) as follows. Let $\mu = (\mu^{(1)}, \ldots, \mu^{(l)}) \in \Pi^l$. Set $k^{(d)}_i = \mu^{(d)}_i + s_d + 1 - i$ for $d = 1, \ldots, l$ and $i \in \mathbb{N}$. Let us write $k^{(d)}_i = c^{(d)}_i - nm^{(d)}_i$ where $c_i \in \{1, \ldots, n\}$, and let $\mathbf{k} = (k_1 > k_2 > \cdots)$ be the ordered sequence whose underlying set is $\{c^{(d)}_i + n(d-1) - nlm^{(d)}_i \,|\, i \in \mathbb{N},\ d = 1, \ldots, l\}$. Let $s = s_1 + \cdots + s_l$. It is easy to see that $k_i = s + 1 - i$ for $i \gg 0$ and we denote by $\zeta(\mu)$ the partition such that $\zeta(\mu)_i = k_i - s + i - 1$. Now let $\mu, \nu \in \Pi^l$. By definition, we set $\mu \leq \nu$ if $\zeta(\mu) \leq \zeta(\nu)$.

A.2 From now on we assume that $v = 1$.

It is more convenient to work with a different basis than $\{\mathbf{f_n}\}$. Let $\mathbf{n} \in \mathcal{M}$ and let $x \in \mathcal{O}_{\mathbf{n}}$. Set $V_k = \mathrm{Ker}\, x^k$ and let $\alpha^1, \dots, \alpha^r \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$ be such that

$$\dim V_k = \alpha^1 + \cdots + \alpha^k, \qquad k = 1, \dots, r$$

and $\dim V_r = \dim \mathbf{n}$. Let $\mathbf{f}_{\alpha^i} \in \mathbf{U}_n^-$ be the characteristic function of the trivial representation of the quiver $\tilde{A}_{n-1}$ on $V_{\alpha^i} \simeq V_i/V_{i-1}$.

Lemma 1 ([VV], Section 13). *We have* $\mathbf{f}_{\alpha^1} \cdots \mathbf{f}_{\alpha^r} \in \mathbf{f_n} + \bigoplus_{\mathbf{l}<\mathbf{n}} \mathbb{N}\mathbf{f_l}$.

Now let $\mathbf{n}, \mathbf{l} \in \mathcal{M}$ such that $\dim \mathbf{n} = \dim \mathbf{l}$. Let (β^k) and (γ^k) be the sequences of dimensions attached as above to $\mathbf{n}$ and $\mathbf{l}$ respectively. If $\mathbf{u}, \mathbf{v} \in \mathbb{Z}/n\mathbb{Z}$ we write $\mathbf{u} \leq \mathbf{v}$ if $\mathbf{u}_i \leq \mathbf{v}_i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Lemma 2. *We have* $\mathbf{n} \geq \mathbf{l}$ *if and only if*

$$\beta^1 + \cdots + \beta^k \leq \gamma^1 + \cdots + \gamma^k \qquad \textit{for all } k. \tag{a}$$

Proof. Straightforward. □

We will write $(\beta^k) \leq (\gamma^k)$ if (a) holds and if $\sum_k \beta^k = \sum_k \gamma^k$. Let $(\alpha^1, \dots, \alpha^r)$ be the sequence attached to $\mathbf{m}$. We will first prove

$$\mathbf{f}_{\alpha^1} \cdots \mathbf{f}_{\alpha^r} \cdot |0, \mathbf{s}_l\rangle \in \mathbb{N}^* |\mathcal{M}_{\mathbf{i}}(\mathbf{p}), \mathbf{s}_l\rangle + \bigoplus_{\mu \not\geq \mathcal{M}_{\mathbf{i}}(\mathbf{p})} \mathbb{N}|\mu, \mathbf{s}_l\rangle \tag{b}$$

$$\mathbf{f}_{\beta^1} \cdots \mathbf{f}_{\beta^r} \cdot |0, \mathbf{s}_l\rangle \in \bigoplus_{\mu \not\geq \mathcal{M}_{\mathbf{i}}(\mathbf{p})} \mathbb{N}|\mu, \mathbf{s}_l\rangle \qquad \textit{for all } (\beta^k) > (\alpha^k). \tag{c}$$

Lemma 3. *Let* $\mu = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi^l$ *and let* $\beta \in N^{\mathbb{Z}/n\mathbb{Z}}$. *We have*

$$\mathbf{f}_\beta \cdot |\mu, \mathbf{s}_l\rangle = \sum_\nu |\nu, \mathbf{s}_l\rangle$$

where the sum ranges over all multipartitions $\nu = (\nu^{(1)}, \dots, \nu^{(l)})$ *such that*

i) $\nu^{(i)} \backslash \mu^{(i)}$ *is a skew diagram with at most one box in each row,*

ii) The number of boxes in $\cup_i \nu^{(i)} \backslash \mu^{(i)}$ *with content* $j \bmod n$ *is* β_j.

Proof. Let $d \in \mathbb{N}^{(\mathbb{Z})}$ such that $d \equiv \beta \pmod{n}$. Then $\gamma_{d|v=1}(\mathbf{f}_\beta) = \overrightarrow{\prod}_i f_i^{(d_i)}$, where $\overrightarrow{\prod}$ denotes the ordered product from $-\infty$ to ∞ (see [VV], Remark 6.1) and where $f_i^{(d_i)}$ is the divided power. Moreover, for any $\sigma \in \Pi^l$,

$$f_i \cdot |\sigma, \mathbf{s}_l\rangle = \sum_\gamma |\gamma, \mathbf{s}_l\rangle$$

where the sum ranges over all $\gamma \in \Pi^l$ such that $\gamma \backslash \sigma$ is an i-box. The Lemma now follows from Section 4.2. □

Finally, recall that $s_1 \gg s_2 \gg \cdots \gg s_l$. It is clear from the definition that for $\mu, \lambda \in \Pi^l$,

$$\mu \geq \lambda \Rightarrow \ \exists\, k \textit{ such that } \mu^{(i)} = \lambda^{(i)} \textit{ for } i = 1, \ldots, k-1 \textit{ and } \mu^{(k)} \geq \lambda^{(k)}. \quad \text{(d)}$$

Note that α_i^k is equal to the number of boxes with content i in the slice s_k of the diagram $D_{\mathbf{p}}$ associated to $\mathbf{p}$. Statements (b) and (c) now easily follow by Lemma 3 and by construction of $\mathcal{M}_{\mathbf{i}}(\mathbf{p})$.

A.3 By [U], Theorem 2.4 it possible to choose $s_l \gg s_{l+1} \gg \cdots \gg s_t$ for some $t \gg 0$ in such a way that $\mathbf{b}_{\mathbf{m}}|0, \mathbf{s}_t\rangle \neq 0$, where $\mathbf{s}_t = (s_1, \ldots, s_t)$.

Lemma 4. *We have* $\mathbf{b}_{\mathbf{m}}|0, \mathbf{s}_t\rangle = \mathbf{b}^+_{\widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p})}$, *where* $\widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p}) = \big(\mathcal{M}_{\mathbf{i}}(\mathbf{p}), 0^{t-l}\big)$.

Proof. By Lemma 1, we have

$$\mathbf{f}_{\alpha^1} \cdots \mathbf{f}_{\alpha^r} \cdot |0, \mathbf{s}_t\rangle \in |\tau_{\mathbf{s}_t}(\mathbf{m}), \mathbf{s}_t\rangle + \bigoplus_{\mu < \tau_{\mathbf{s}_t}(\mathbf{m})} \mathbb{Z}|\mu, \mathbf{s}_t\rangle.$$

But from (b) and (d) it is clear that

$$\mathbf{f}_{\alpha^1} \cdots \mathbf{f}_{\alpha^r} \cdot |0, \mathbf{s}_t\rangle \in |\widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p}), \mathbf{s}_t\rangle + \bigoplus_{\mu \not\geq \widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p})} \mathbb{Z}|\mu, \mathbf{s}_t\rangle.$$

Hence $\tau_{\mathbf{s}_t}(\mathbf{m}) = \widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p})$. $\square$

In particular,

$$\mathbf{b}_{\mathbf{m}} \cdot |0, \mathbf{s}_t\rangle \in |\widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p}), \mathbf{s}_t\rangle + \bigoplus_{\mu < \widetilde{\mathcal{M}}_{\mathbf{i}}(\mathbf{p})} \mathbb{N}|\mu, \mathbf{s}_l\rangle.$$

Consider the projection $\pi : \Lambda^\infty_{\mathbf{s}_t} \to \Lambda^\infty_{\mathbf{s}_l}$ given by

$$|\big(\mu^{(1)}, \ldots, \mu^{(t)}\big), \mathbf{s}_t\rangle \mapsto \begin{cases} |\big(\mu^{(1)}, \ldots, \mu^{(l)}\big), \mathbf{s}_l\rangle & \textit{if } \mu^{(j)} = 0 \textit{ for } j > l \\ 0 & \textit{otherwise} \end{cases}$$

It is clear from (4.1) that $\pi(\mathbf{b}_{\mathbf{m}} \cdot |0, \mathbf{s}_t\rangle) = \mathbf{b}_{\mathbf{m}} \cdot |0, \mathbf{s}_l\rangle$. Hence

$$\mathbf{b}_{\mathbf{m}} \cdot |0, \mathbf{s}_l\rangle \in |\mathcal{M}_{\mathbf{i}}(\mathbf{p}), \mathbf{s}_l\rangle + \bigoplus_{\mu < \mathcal{M}_{\mathbf{i}}(\mathbf{p})} \mathbb{N}|\mu, \mathbf{s}_l\rangle.$$

This proves Lemma 5.4 $\square$

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